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Integral Representation and Embedding of Weak Markov Systems

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In the sequel, A will always denote a subset of the real line having at least $n+2$ elements $(n \geqslant 0)$, $l_1 = \inf(A)$, $l_2 = \sup(A)$, and $I(A)$ will denote the convex hull of A (thus for example, if $A = [2, 3) \cup (4, \infty)$, then $I(A) = [2, \infty))$; $Z_n = \{z_0, ..., z_n\}$ will be a set of real valued functions defined on A; by $S(Z_n)$ we shall denote the linear span of Z_n. We shall call Z_n a weak Čebyšev system (Čebyšev system), provided that Z_n be linearly independent on A, and for every choice of $n+1$ points t_i of A, with $t_0 < t_1 < \cdots < t_n$, $\det[z_i(t_j);\ i, j = 0, ..., n] \geqslant 0$ (>0). If Z_k is a (weak) Čebyšev system for $k = 0, ..., n$, then Z_n will be called a (weak) Markov system. A normalized—or normed—(weak) Markov system is a (weak) Markov system Z_n for which $z_0 \equiv 1$. Markov systems are also called complete Čebyšev systems (cf. Karlin and Studden [2]). We shall say that $U_n = \{u_0, ..., u_n\}$ has been obtained from Z_n by a triangular linear transformation if $u_0 = z_0$, and

$$u_k - z_k \in S(Z_{k-1}), \qquad k = 1, 2, ..., n.$$

Note that if Z_n is linearly independent then, for $k = 0, 1, ..., n$, U_k is a basis of $S(Z_k)$. We shall adopt the convention that if $b \leqslant a$, then $[a, b) = (a, b] = \varnothing$.

In [6, Theorem 1] we gave an integral representation of Markov systems. Recently Zielke [11] gave a counterexample and a corrected version of this result, and generalized it to a class of normalized weak Markov systems. The purpose of our paper is to extend the results of [11], using a refinement of a new embedding property of normalized weak Markov systems developed in [7].

A system Z_n will be called nondegenerate if for every c in A, Z_n is linearly independent both on $(-\infty, c) \cap A$ and on $(c, \infty) \cap A$, and it will be called weakly nondegenerate provided that the following conditions are satisfied:

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Condition I. For every real number c, Z_n is linearly independent on at least one of the sets $(c, \infty) \cap A$ and $(-\infty, c) \cap A$.

Condition E. For every point c in $I(A)$ we have:

(a) If Z_n is linearly independent on $[c, \infty) \cap A$, then there exists a set U_n, obtained from Z_n by a triangular linear transformation, such that for any sequence $\{k(r);\ r=0, ..., m\}$ with $k(0) \geqslant 0$ and $k(m) \leqslant n$ that is either strictly increasing or contains exactly one element, the set $\{u_{k(r)};\ r=0, ..., m\}$ is a weak Markov system on $[c, \infty) \cap A$.

(b) If Z_n is linearly independent on $(-\infty, c] \cap A$, then there exists a set V_n, obtained from Z_n by a triangular linear transformation, such that for every sequence $\{k(r);\ r=0, ..., m\}$ with $k(0) \geqslant 0$ and $k(m) \leqslant n$ that is either strictly increasing or contains exactly one element, $\{(-1)^{r-k(r)} v_{k(r)};\ r=0, ..., m\}$ is a weak Markov system on $(-\infty, c] \cap A$.

Finally Z_n will be called "representable" if for any point c in A there exist a set $U_n = \{u_0, ..., u_n\}$, obtained from Z_n by a triangular linear transformation; a strictly increasing and bounded real function $h(t)$, defined on A and such that $h(c) = c$; and continuous, increasing, and nonconstant real functions $w_k(t)$, defined on $I(h(A))$, such that for all x in A

$$\begin{aligned} u_0 &\equiv 1 \\ u_1(x) &= \int_c^{h(x)} dw_1(t_1) \\ &\vdots \\ u_n(x) &= \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{n-1}} dw_n(t_n) \cdots dw_1(t_1). \end{aligned} \tag{1}$$

In [11], Zielke essentially proved that a nondegenerate normalized weak Markov system is representable. Our main result is:

THEOREM 1. *Every weakly nondegenerate normalized weak Markov system is representable.*

Remarks. (i) In the statement of [11, Theorem 3], Zielke asserts that the representation (1) is valid for *some* point c, but in the proof of the theorem he actually shows that a representation exists for *any* point c in A. The distinction is, however, immaterial: If (1) is satisfied for some point c in A it is easy to see that for any other point c' in A there is a basis U'_n of $S(Z_n)$, obtained from U_n by a triangular linear transformation, having a representation of the form (1) with c replaced by c'.

(ii) It can be shown that every nondegenerate normalized weak Markov system satisfies Condition E (this has essentially been done in the proof of

[8, Theorem 2]). Thus, every nondegenerate normalized weak Markov system is weakly nondegenerate. The converse, however, is false: Let $u_0 \equiv 1$, $u_1(x) = x$ on $(0, 1]$, $u_1(x) = 1$ on $[1, 2)$, and $u_2(x) = [u_1(x)]^2$ on $(0, 2)$. Then $U_2 = \{u_0, u_1, u_2\}$ is a weakly nondegenerate normalized weak Markov system on $(0, 2)$. However U_2 is not nondegenerate there.

(iii) The converse of Theorem 1 is false: let $h(t) = t$, $w_1(t) = 1$ on $(-1, 0)$, $w_1(t) = t + 1$ on $[0, 1)$, $w_2(t) = t$ on $(-1, 0)$, and $w_2(t) = 0$ on $[0, 2)$. If $U_2 = \{u_0, u_1, u_2\}$ has a representation of the form (1), it is readily seen that $u_2 \equiv 0$ on $(-1, 1)$.

We shall call Z_n strongly representable if it is representable and all the functions $w_i(t)$ are strictly increasing on $h(A)$. We shall say that A has property (B), if for any two elements of A there is a third element of A between them. As a consequence of Theorem 1 we shall prove:

THEOREM 2. *Let A have property* (B) *and assume that Z_n is weakly nondegenerate. Then Z_n is a normalized Markov system if and only if it is strongly representable.*

COROLLARY. *Let A have property* (B) *and assume that Z_n is weakly nondegenerate. Then if Z_n is a normalized Markov system on A there is a function z_{n+1} such that also $Z_n \cup \{z_{n+1}\}$ is a normalized Markov system on A.*

If Z_n is a Markov system, it is obvious that Condition I is satisfied. If, moreover, A satisfies property (B), it is easy to see that Condition E is satisfied for any point in $(l_1, l_2) \cap A$, making the assumption of weak nondegeneracy redundant if neither l_1 nor l_2 are in A; thus this corollary generalizes the main result of [9]. Since it is not known as yet under what circumstances Condition E will be satisfied at an endpoint, it is at present unclear whether the corollary also generalizes the main result of [5]. We intend to study this problem in a later paper.

A system Z_n is called C-bounded if every element of Z_n is bounded on the intersection of A with any compact subset of $I(A)$; if A is an interval and every element of Z_n is absolutely continuous in any closed subinterval of A, we shall say that Z_n is C-absolutely continuous. If $V_n = \{v_0, ..., v_n\}$ is a set of real functions defined on a real set B we say that Z_n can be embedded in V_n if there is a strictly increasing function $h: A \to B$ such that $v_i[h(t)] = z_i(t)$ for every $t \in A$ and $i = 0, 1, ..., n$. The function h is called an embedding function.

In the proof of Theorem 1 we shall need the following refinement of the theorem of [7]:

THEOREM 3. *Let c be an element of A. If Z_n is a weakly nondegenerate normalized weak Markov system on A, then Z_n can be embedded in a weakly nondegenerate normalized weak Markov system V_n of C-absolutely con-*

tinuous functions defined on an open bounded interval, and V_n and the embedding function $h(t)$ can be chosen so that $h(c)=c$. Moreover if A satisfies property (B), *the converse statement is also true.*

The proof of Theorem 3 is based on the following auxiliary propositions:

LEMMA 1. *Let Z_n be a weakly nondegenerate weak Markov system on a set A, let $p: A \to R$ be a strictly increasing function, and let $v_r(t) = z_r(p^{-1}(t))$, $r=0, ..., n$. Then V_n is a weakly nondegenerate weak Markov system on $p(A)$.*

The proof of Lemma 1 is straightforward and will be omitted.

LEMMA 2. *Let $[a, b]$ be a closed bounded interval. Assume that f is a continuous function of bounded variation and that g is a strictly increasing continuous function, both defined on $[a, b]$. For $a \leqslant \alpha \leqslant \beta \leqslant b$, let $V(f, \alpha, \beta)$ denote the total variation of f on $[\alpha, \beta]$. Let $c \in [a, b]$ be arbitrary but fixed, and define $v(f, t)$ to equal $V(f, c, t)$ on $[c, b]$ and $-V(f, t, c)$ on $[a, c)$. Finally, let $q(t) = g(t) + v(f, t)$ and $h(t) = f[q^{-1}(t)]$. Then $h(t)$ is absolutely continuous on $[q(a), q(b)]$.*

Proof of Lemma 2. The hypotheses imply that $q(t)$ is strictly increasing and continuous; thus $q^{-1}(t)$ is strictly increasing on $[q(a), q(b)]$. If $a < s_1 < s_2 < b$, then $|f(s_2) - f(s_1)| \leqslant V(f, s_1, s_2) = v(f, s_2) - v(f, s_1) \leqslant v(f, s_2) - v(f, s_1) + g(s_2) - g(s_1) = q(s_2) - q(s_1)$. Thus, if (α_1, β_1), (α_2, β_2), ..., (α_n, β_n) are disjoint subintervals of $[q(a), q(b)]$ we have

$$\sum_{i=1}^{n} |h(\beta_i) - h(\alpha_i)| = \sum_{i=1}^{n} |f[q^{-1}(\beta_i)] - f[q^{-1}(\alpha_i)]|$$

$$\leqslant \sum_{i=1}^{n} (q[q^{-1}(\beta_i)] - q[q^{-1}(\alpha_i)]) = \sum_{i=1}^{n} (\beta_i - \alpha_i),$$

and the conclusion follows. Q.E.D.

The following lemma implies that every weakly nondegenerate normalized weak Markov system is C-bounded:

LEMMA 3. *Let $U_n = \{u_0, ..., u_n\}$ be a weakly nondegenerate normalized weak Markov system on a set A, let $l_1 = \inf(A)$, $l_2 = \sup(A)$, $c \in I(A)$, and let u be any function in $S(U_n)$.*

(a) *If $c > l_1$ and c is a point of accumulation of $(l_1, c) \cap A$, then $\lim_{t \to c^-} u(t)$ exists and is finite.*

(b) *If $c < l_2$ and c is a point of accumulation of $(c, l_2) \cap A$, then $\lim_{t \to c^+} u(t)$ exists and is finite.*

Proof. We only prove (a); the proof of (b) is similar and will be omitted.

We proceed by induction. The assertion is trivially true for $n=0$. To prove the inductive step, assume that for any function w in $S(U_{n-1})$ (where $U_{n-1}=\{u_0, ..., u_{n-1}\}$), $\lim_{t \to c^-} w(t)$ exists and is finite. If U_n is linearly independent on $(-\infty, c) \cap A$ there is a number $d \in A$ such that $d \geqslant c$. Indeed, this is obvious if $c < l_2$, whereas if $c = l_2$ we can take $d = l_2$. Since clearly U_n is linearly independent on $(-\infty, d] \cap A$, by Condition E we conclude that there is a function $u = u_n + w$, with $w \in S(U_{n-1})$, such that u is monotonic on $(-\infty, d] \cap A$, whence the conclusion readily follows. Assume now that U_n is linearly dependent on $(-\infty, c) \cap A$. Condition I then implies that U_n is inearly independent on $(c, \infty) \cap A$, and therefore on any set of the form $(d', \infty) \cap A$, $d' < c$. Another application of Condition E readily yields the conclusion for this case as well. Q.E.D.

The proof of the next proposition was sketched in [7].

LEMMA 4. *Let Z_n be a normalized weak Markov system of bounded functions defined on a closed interval $I = [a, b]$. Then all the elements of $S(Z_n)$ are of bounded variation on I.*

Proof. Let z be a function in $S(Z_n)$, arbitrary but fixed, let γ be any real number, and let $v(\gamma)$ denote the number of sign changes of $z(t) - \gamma$. Since [10, p. 12, Lemma 4.1] implies that $v(\gamma) \leqslant n$, and the boundedness of $z(t)$ implies that $v(\gamma)$ has bounded support, the conclusion follows from, e.g., [4, p. 257, Theorem 6]. Q.E.D.

LEMMA 5. *Let Z_n be a weakly nondegenerate normalized weak Markov system defined on an interval I (open, closed, or semiopen), and let $c \in I$. If z_1 is continuous at c, then all the elements of $S(Z_n)$ are continuous at c.*

Proof. We shall only prove that if $c > \inf(I)$, then all the elements of $S(Z_n)$ are left-continuous at c. The proof of the other case is similar and will be omitted.

We proceed by induction on n. For $n=1$ the assertion is true by hypothesis; assume therefore that $n > 1$.

If Z_n is linearly independent on $S_1 = (-\infty, c] \cap I$, then from Condition E we readily conclude that there is a set U_n, obtained from Z_n by a triangular linear transformation, such that both $\{1, (-1)^{n-1} u_n\}$ and $\{1, u_1, (-1)^n u_n\}$ are weak Markov systems on S_1. The first assertion is equivalent to saying that $(-1)^{n-1} u_n$ is increasing on S_1, from which we conclude that $(-1)^{n-1} u_n(c^-) \leqslant (-1)^{n-1} u_n(c)$. The linear independence implies that there is a point t_0 in $(-\infty, c) \cap I$ such that $u_1(t_0) < u_1(c)$.

Assume that $t_0 < t < c$; then, subtracting the second column from the third, we have

$$0 \leqslant \begin{vmatrix} 1 & 1 & 1 \\ u_1(t_0) & u_1(t) & u_1(c) \\ (-1)^n u_n(t_0) & (-1)^n u_n(t) & (-1)^n u_n(c) \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} 1 & 1 & 0 \\ u_1(t_0) & u_1(t) & u_1(c) - u_1(t) \\ u_n(t_0) & u_n(t) & u_n(c) - u_n(t) \end{vmatrix} .$$

Since $u_1(t)$ is continuous at c, passing to the limit we have

$$0 \leqslant (-1)^n \begin{vmatrix} 1 & 1 & 0 \\ u_1(t_0) & u_1(c) & 0 \\ u_n(t_0) & u_n(c^-) & u_n(c) - u_n(c^-) \end{vmatrix}$$

$$= (-1)^n [u_1(c) - u_1(t_0)][u_n(c) - u_n(c^-)],$$

whence we conclude that $(-1)^{n-1} u_n(c) \leqslant (-1)^{n-1} u_n(c^-)$. We have therefore shown that $u_n(t)$ is left-continuous at c. Since $u_n = z_n + w$, with $w \in S(Z_{n-1})$, applying the inductive hypothesis we conclude that also z_n is left-continuous at c.

If Z_n is linearly dependent on $(-\infty, c] \cap I$, from Condition I we conclude that Z_n must be linearly independent on $(c, \infty) \cap I$. Thus, if d is an arbitrary but fixed point in $(-\infty, c) \cap I$, it is clear that Z_n is linearly independent on $J_2 = [d, \infty) \cap I$, whence by Condition E there is a set V_n, obtained from Z_n by a triangular linear transformation, such that both $\{1, v_n\}$ and $\{1, v_1, v_n\}$ are normalized weak Markov systems on J_2. The first assertion is equivalent to saying that v_n is increasing on J_2, from which we conclude that $v_n(c^-) \leqslant v_n(c)$. The linear independence implies that there is a point t_1 in $(c, \infty) \cap I$ such that $v_1(c) < v_1(t_1)$. Choosing $t < c$ and proceeding as in the preceding paragraph, we deduce that $[v_n(c^-) - v_n(c)][v_1(t_1) - v_1(c)] \geqslant 0$, whence $v_n(c) \leqslant v_n(c^-)$, and the conclusion readily follows. Q.E.D.

Proof of Theorem 3. Assume that Z_n is a weakly nondegenerate normalized weak Markov system.

From [7] we know that Z_n can be embedded in a normalized weak Markov system $U_n = \{u_0, ..., u_n\}$ of continuous functions defined on an open interval (a_1, b_1), and such that if h is the embedding function then $h(c) = c$.

Assuming now that Z_n is weakly nondegenerate, we shall adapt the procedure outlined in the proof of the theorem of [7] to show that U_n is also weakly nondegenerate.

Let $S=\{s_i\}$ denote the set of points of accumulation of A at which $z_1(t)$ has jump discontinuities. If $s_i \in S \cap (l_1, l_2)$, let $d_i = 2^{-i}$; on the other hand if $s_i \in A$, let $a_i = 2^{-(i+1)}$ if $z_1(s_i^+) - z_1(s_i) \neq 0$, and 0 otherwise. If $s_i = l_1$ and $l_1 \in A$, let $d_i = a_i = |z_i(s_i^+) - z_i(s_i)|$, whereas if $s_i = l_2$ and $l_2 \in A$, we define $a_i = |z_1(s_i) - z_1(s_i^-)|$. Let $q(t) = t + \sum_{s_j < t} d_j$ if $t \in A$ but $t \notin S$, whereas for $t_i \in A \cap S$ we define $q(t_i) = t_i + (\sum_{s_j < t_i} d_j) + a_i$. It is clear that $q(t)$ is strictly increasing. (Note that there is a typographical error in the definitions of α_i and β_i in [7]. They should be defined in a manner similar to that of a_i above.)

Setting $z_k^{(0)}(t) = z_k[q^{-1}(t)]$, we infer from Lemma 1 that $Z_n^{(0)}$ is weakly nondegenerate on $A^{(0)} = q(A)$. Moreover, it has the property that $z_1^{(0)}$ is either continuous or has a removable discontinuity at every point of accumulation of $A^{(0)}$.

Let $l_1^{(0)} = \inf(A^{(0)})$, $l_2^{(0)} = \sup(A^{(0)})$. If $l_1^{(0)}$ belongs to $A^{(0)}$, define $z_r^{(1)}$ to equal $z_r^{(0)}(l_1)$ on $(-\infty, l_1^{(0)})$; if $l_2^{(0)}$ belongs to $A^{(0)}$, define $z_r^{(1)}$ to equal $z_r^{(0)}(l_2^{(0)})$ on $(l_2^{(0)}, \infty)$; moreover, let $z_r^{(1)} = z_r^{(0)}$ on $A^{(0)}$. Clearly $Z_n^{(1)}$ is a weakly nondegenerate normalized weak Markov system defined on a set $A^{(1)}$ that has no first nor last element. Let $l_1^{(1)} = \inf(A^{(1)})$, $l_2^{(1)} = \sup(A^{(1)})$, and let $\bar{A}^{(1)}$ denote the closure of $A^{(1)}$ in the relative topology of $I = (l_1^{(1)}, l_2^{(1)})$. If x is in $\bar{A}^{(1)}$ but not in $A^{(1)}$, define $z_r^{(2)}(x) = \lim_{t \to x^-} z_r^{(1)}(t)$, $r = 0, ..., n$, if x is a point accumulation of $(-\infty, x) \cap A^{(1)}$, or $z_r^{(2)}(x) = \lim_{t \to x^+} z_r^{(1)}(t)$, $r = 0, ..., n$, if it is not (this can be done because of Lemma 3), whereas for x in $A^{(1)}$, let $z_r^{(2)}(x) = z_r^{(1)}(x)$. Clearly $Z_n^{(2)}$ is a normalized weak Markov system on $\bar{A}^{(1)}$.

It is readily seen that $Z_n^{(2)}$ is weakly nondegenerate on $\bar{A}^{(1)}$: To prove Condition I for $Z_n^{(2)}$, assume, e.g., that $Z_n^{(2)}$ is linearly dependent on $[c, \infty) \cap \bar{A}^{(1)}$. From Condition I for $Z_n^{(1)}$ we readily infer that $Z_n^{(1)}$ is linearly independent on $(-\infty, c] \cap A^{(1)}$, which clearly implies that $Z_n^{(2)}$ is linearly independent on $(-\infty, c] \cap \bar{A}^{(1)}$. To prove Condition E, assume, for example, that $Z_n^{(2)}$ is linearly independent on $[c, \infty) \cap \bar{A}^{(1)}$. Let $d \in A^{(1)}$, $d < c$ be arbitrary but fixed. Since $Z_n^{(1)}$ is clearly linearly independent on $[d, \infty) \cap A^{(1)}$, applying Condition E to $Z_n^{(1)}$ on $[d, \infty) \cap A^{(1)}$ and passing to the limit, Condition E for $Z_n^{(2)}$ on $[c, \infty) \cap \bar{A}^{(1)}$ readily follows.

Clearly the complementary set of $\bar{A}^{(1)}$ in $(l_1^{(1)}, l_2^{(1)})$, if not empty, is a disjoint union of open intervals V_j; moreover if $c_j = \inf(V_j)$ and $d_j = \sup(V_j)$, then both c_j and d_j are in $\bar{A}^{(1)}$. Let w_r be defined on I as follows: If $t \in \bar{A}^{(1)}$, then $w_r(t) = z_r^{(2)}(t)$. On the other hand, if $t \notin \bar{A}^{(1)}$, then $c_i < t < d_i$ for some i. In this case, define $w_r(t) = [(d_i - t) z_r^{(2)}(c_i) + (t - c_i) z_r^{(2)}(d_i)]/(d_i - c_i)$. (Note that $t = [(d_i - t)c_i + (t - c_i)d_i]/(d_i - c_i)$.) It is readily seen that $W_n = \{w_0, ..., w_n\}$ is a normalized weakly nondegenerate weak Markov system defined on the open interval I. Since $z_1^{(2)}$ is clearly continuous on $\bar{A}^{(1)}$, and w_1 is obtained from it by linear interpolation, we readily deduce that w_1 is continuous on I. Applying Lemma 5, we thus conclude that all the elements of W_n are continuous on I. We have therefore shown that Z_n can be embedded in a weakly nondegenerate normalized weak Markov system W_n of continuous functions defined on an open interval I. Moreover, from Lemma 4 we know that the elements of $S(W_n)$ are of bounded variation on every closed subinterval of I. Thus, if the functions $v(w_k, t)$ are defined as in Lemma 2, $p(t) = t + \sum_{k=1}^{n} v(w_k, t)$, $I_1 = p(I)$, $w_i^{(1)}(t) = w_i[p^{-1}(t)]$, $i = 0, 1, 2, ..., n$, and $W_n^{(1)} = \{w_0^{(1)}, w_1^{(1)}, ..., w_n^{(1)}\}$, it is readily seen from Lemmas 1 and 2 that $W_n^{(1)}$ is a weakly nondegenerate normalized weak Markov system of C-absolutely continuous functions on I_1. Thus, there is a strictly increasing function $h(t)$ that embeds Z_n into $W_n^{(1)}$. Setting $p_1(t) = c - h(c) + t$, $h_1(t) = p_1[h(t)]$, $w_i^{(2)}(t) = w_i^{(1)}[p_1^{-1}(t)]$, and $W_n^{(2)} = \{w_0^{(2)}, ..., w_n^{(2)}\}$, it is easy to see that $h_1(t)$ embeds Z_n into $W_n^{(2)}$, and that $h_1(c) = c$. Making if necessary a change of variable of the form $c + \arctan(t - c)$ to ensure the boundedness of the domain of the elements of $W_n^{(2)}$, the conclusion readily follows.

The proof of the converse is trivial and will be omitted. Q.E.D.

To prove Theorem 1 we also need the following:

LEMMA 6. *Let* $U_n = \{u_0, ..., u_n\}$ *be a weakly nondegenerate weak Markov system on an interval* (a, b). *If for some* c *in* (a, b), $u_0(c) = 0$, *then* $u_k(c) = 0$, $k = 1, 2, ..., n$.

Proof. We proceed by induction on n. For $n = 0$ the assertion is true by hypothesis. To prove the inductive step, assume first that U_n is linearly independent on (a, c). Then from Conditon E we readily conclude that there is a set V_n, obtained from U_n by a triangular linear transformation, such that both $\{(-1)^n v_n\}$ and $\{v_0, (-1)^{n+1} v_n\}$ are weak Markov systems on $(a, c]$. From the first condition we conclude that $(-1)^n v_n(c) \geqslant 0$. The linear independence implies that there is a point $t_0 \in (a, c)$, such that $v_0(t_0) \neq 0$. Applying the second condition we readily deduce that $v_0(t_0) > 0$ and that $v_0(t_0)(-1)^{n+1} v_n(c) \geqslant 0$. Thus $(-1)^n v_n(c) \leqslant 0$, and the assertion readily follows.

If U_n is linearly dependent on (a, c), from Condition I we deduce that it must be linearly independent on (c, b), and the assertion is proved by a similar procedure. Q.E.D.

Proof of Theorem 1. Let Z_n be a weakly nondegenerate normalized weak Markov system. Without loss of generality we can assume that $z_i(c) = 0$, $i = 1, ..., n$. From Theorem 3 we know that there is a strictly increasing function $p: A \to (a_1, b_1)$ and a weakly nondegenerate normalized weak Markov system $\{q_0, ..., q_n\}$ of C-absolutely continuous functions defined on (a_1, b_1), such that $z_i = q_i \circ p$, $i = 0, ..., n$, and $p(c) = c$. Clearly $q_i(c) = 0$, $i = 1, ..., n$; moreover, if D is the set of points on which the functions q_i are differentiable, then the measure of D equals $b_1 - a_1$. Let I be a subinterval of (a_1, b_1), and let $\{k(r);\ r=0, ..., m\}$ be a strictly increasing sequence with $k(0)=0$, $k(1)\geqslant 1$, and $k(m)\leqslant n$. Since the functions $q_i(t)$ are C-absolutely continuous, it is readily seen that $\{q'_{k(r)};\ r=1, ..., m\}$ is linearly dependent on $I\cap D$ if and only if $\{q_{k(r)};\ r=0, ..., m\}$ is linearly dependent on I. Thus, proceeding as in [10, Theorem 11.3(b)] we readily infer that $Q'_{n-1}=\{q'_1, ..., q'_n\}$ is a weakly nondegenerate weak Markov system on D; thus if S is the subset of D on which $q'_1\neq 0$, and $m_i = q'_i/q'_1$, from Lemma 6 we readily deduce that $M_{n-1}=\{m_1, ..., m_n\}$ is a weakly nondegenerate normalized weak Markov system on S.

Let $a_2=\inf(S)$, $b_2=\sup(S)$, and assume for instance that $a_2>a_1$ and $b_2<b_1$. This implies that $q'_1(t)=0$ on $(a_1, a_2)\cap D$ and on $(b_2, b_1)\cap D$. In particular, Condition I implies that Q'_{n-1} is linearly independent on $(a_1, b_2]\cap D$. Thus Condition E implies that there is a system $R_{n-1}=\{r_1, ..., r_n\}$, obtained from Q'_{n-1} by a triangular linear transformation, such that $\{(-1)^{i-1}r_i\}$ and $\{r_1, (-1)^i r_i\}$, $i=2, ..., n$, are weak Markov systems on D. Since $r_1\equiv q'_1>0$ on S, this means that $(-1)^i r_i/r_1$ is both increasing and nonpositive (and therefore bounded from above) on S. Setting $u_i=r_i/r_1$ on S and $u_i(t)=\lim_{t\to b_2^-} r_i(t)/r_1(t)$ on $[b_2, b_1)$, it is clear that $U_n=\{u_1, ..., u_n\}$ is a weakly nondegenerate weak Markov system on $S\cup[b_2, b_1)$. This means that there is a weakly nondegenerate normalized weak Markov system $M^{(0)}_{n-1}=\{m^{(0)}_1, ..., m^{(0)}_n\}$ on $S\cup[b_2, b_1)$ that coincides with M_{n-1} on S, and such that the functions $m^{(0)}_k$ are constant on $[b_2, b_1)$. Applying Condition E again and using a similar procedure, it is easy to see that $M^{(0)}_{n-1}$ can be extended to the left; i.e., there exists a weakly nondegenerate normalized weak Markov system $M^{(1)}_{n-1}=\{m^{(1)}_1, ..., m^{(1)}_n\}$ that coincides with M_{n-1} on S, and such that the functions $m^{(1)}_k$ are constant on $(a_1, a_2]$ and on $[b_2, b_1)$. Proceeding as in the proof of Theorem 3 we readily see that there is a weakly nondegenerate normalized weak Markov system $V_{n-1}=\{v_1, ..., v_n\}$ on (a_1, b_1) that coincides with M_{n-1} on S. Since Lemma 6 implies that all the functions q'_i vanish on $D-S$, we conclude that $q'_i(t)=q'_1(t)\,v_i(t)$ for every t in D and $i=1, ..., n$. It is therefore clear that for every x in (a_1, b_1),

$$q_i(x)=\int_c^x q'_1(t)\,v_i(t)\,dt, \qquad i=1, ..., n. \tag{2}$$

The proof is completed by induction. For $n=1$ the assertion of the theorem follows from (2). We now proceed to the proof of the inductive step.

By inductive hypothesis there is a basis $\{\bar{v}_1, ..., \bar{v}_n\}$ of the linear span of $\{v_1, ..., v_n\}$, such that for $i=1, ..., n$ and $x\in(a_1, b_1)$, $\bar{v}_i(x)=p_i[h(x)]$, where

$$p_i(x)=\int_c^x\int_c^{t_2}\cdots\int_c^{t_{i-1}} dw_i(t_i)\cdots dw_2(t_2),$$

$h(x)$ is strictly increasing and bounded on (a_1, b_1), $h(c)=c$, and the functions w_i are continuous and increasing on $(h(a_1^+), h(b_1^-))$. There is no loss of generality if we assume that $\bar{v}_i = v_i$, $i=1, ..., n$. It is clear that the inverse function of h can be extended to an increasing (but not necessarily strictly increasing) function g, continuous on $(h(a_1^+), h(b_1^-))$; thus since the functions $p_i(x)$ are continuous, setting $w_1(x) = \int_c^{g(x)} q_1'(t)\,dt$ and applying [1, p. 182, Lemma 8(f); 3, p. 368, Theorem 1] we easily conclude that $q_i(x) = \int_c^{h(x)} p_i(t)\,dw_1(t)$, whence the assertion readily follows. Q.E.D.

Proof of Theorem 2. From Theorem 1 we know that Z_n is representable. Let $U_n = \{u_0, ..., u_n\}$ be a basis of $S(Z_n)$ having the representation (1). One easily sees (as in [5, Lemma 2]) that U_n is a normalized Markov system on A.

Assume that for some k, $w_k(t)$ is constant on some subinterval I of $I(h(A))$ that contains two points of $h(A)$. By an inductive procedure involving the number of integrations we see that $u_k[h^{-1}(t)]$ can be expressed as a linear combination of $u_0(h^{-1}(t)), ..., u_{k-1}(h^{-1}(t))$ on I. Thus u_k can be expressed as a linear combination of $u_0, ..., u_{k-1}$ on $h^{-1}(I)$. Since h is strictly increasing and A has property (B), $h^{-1}(I)$ has an infinite number of points. Since U_k is a Čebyšev system we have obtained a contradiction.

To prove the converse, let U_n be a basis having a representation of the form (1), where the functions h and w_i satisfy the hypotheses of the theorem. For $k=0, ..., n$, let $v_k(x) = u_k[h^{-1}(x)]$; it suffices to prove that V_n is a normalized Markov system on $h(A)$. Since h is strictly increasing, it is clear $h(A)$ has property (B). Thus, if $\{x_i; i=0, ..., n\}$ is an arbitrary subset of $h(A)$, with $x_0 < x_1 < \cdots < x_n$, there is a subset $\{t_i;\ i=1, ..., n\}$ of $h(A)$ with $x_{i-1} < t_i < x_i$. We now proceed by induction. The assertion is clearly trivial for $n=0$ and $n=1$. To prove the inductive step, let $a = \inf(h(A))$, $b = \sup(h(A))$, and let $f_r(t)$ be defined as follows: $f_1 = 1$, $f_2(t) = \int_c^t dw_2(t_1)$, and for $r=3, ..., n$, $f_r(t) = \int_c^t \int_c^{t_2} \cdots \int_c^{t_{r-1}} dw_r(t_r) \cdots dw_2(t_2)$. Clearly $v_r(x) = \int_c^x f_r(t)\,dw_1(t)$, $r=1, ..., n$. By inductive hypothesis $\{f_1, ..., f_n\}$ is a normalized Markov system on $h(A)$. In particular, this implies that for every k, $k=1, ..., n$, $\det[f_i(t_j);\ i, j=1, ..., k] > 0$. By continuity we conclude that for each i there is a subinterval J_i of (x_{i-1}, x_i) such that if $s_i \in J_i$ for each i, then for each k, $k=1, ..., n$, $\det[f_i(s_j);\ i, j=1, ..., k] > 0$. Proceeding as in, e.g., the proof of [2, p. 382, Lemma 1], we see that for any k, $k=0, 1, ..., n$,

$$\det[v_i(x_j);\ i, j=0, ..., k]$$

$$= \int_{x_0}^{x_1} \int_{x_1}^{x_2} \cdots \int_{x_{k-1}}^{x_k} \det[f_i(s_j);\ i, j=1, ..., k]\, dw_1(s_k)\, dw_1(s_{k-1}) \cdots dw_1(s_1).$$

Since, moreover, $\{f_1, ..., f_n\}$ is a weak Markov system on (a, b), the conclusion readily follows. Q.E.D.

Proof of corollary. From Theorem 2 there is a basis U_n of Z_n having a representation of the form (1), where $h(t)$ is strictly increasing, and the $w_i(t)$ are increasing on $I(h(A))$, and strictly increasing on $h(A)$. Setting $w_{n+1}(x)=x$ and

$$z_{n+1}(x)=\int_c^{h(x)}\int_c^{t_1}\cdots\int_c^{t_n} dw_{n+1}(t_{n+1})\cdots dw_1(t_1),$$

we readily obtain the conclusion. Q.E.D.

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